Interpolation & Polynomial Approximation
Divided Differences

Outline

1. Introduction to Divided Differences
2. The Divided Difference Notation
3. Newton's Divided Difference Interpolating Polynomial
4. Example and Matlab program

Introduction to Divided Differences

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. Although this polynomial is unique, there are alternate algebraic representations that are useful in certain situations. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0) \cdots (x - x_{n-1}), \quad (3.5)$$

for appropriate constants $a_0, a_1, \ldots, a_n$. To determine the first of these constants, a_0, note that if $P_n(x)$ is written in the form of Eq. (3.5), then evaluating $P_n(x)$ at x_0 leaves only the constant term a_0; that is,

$$a_0 = P_n(x_0) = f(x_0).$$

Introduction to Divided Differences

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P(x)$ is evaluated at x_1, the only nonzero terms in the evaluation of $P_n(x_1)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1);$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (3.6)$$

Introduction to Divided Differences

We now introduce the divided-difference notation, which is related to Aitken's Δ^2 notation used in Section 2.5. The *zeroth divided difference* of the function f with respect to x_i, denoted $f[x_i]$, is simply the value of f at x_i:

$$f[x_i] = f(x_i). \tag{3.7}$$

The remaining divided differences are defined recursively; the *first divided difference* of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}. \tag{3.8}$$

Introduction to Divided Differences

The *second divided difference*, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

Similarly, after the $(k-1)$st divided differences,

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}] \quad \text{and} \quad f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}],$$

have been determined, the ***k*th divided difference** relative to $x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}. \tag{3.9}$$

The process ends with the single *nth divided difference*,

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Divided Differences Table

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

Introduction to Divided Differences

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

$$a_0 = f(x_0) = f[x_0]. \qquad a_1 = f[x_0, x_1], \qquad a_k = f[x_0, x_1, x_2, \ldots, x_k],$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}). \qquad (3.10)$$

Newton’s Divided-Difference Formula

Introduction to Divided Differences

Example 1

The following table lists values of a function *f* at various points.

x	0.6	1.0	1.2	1.4
f	0.36	3.00	5.76	9.80

a) Use Newton's Divided-Difference Formula of degrees one, two, and three to approximate $f(1.1)$.

b) Find the absolute error if $f(x) = 5x^3 - 2x^2$.

Introduction to Divided Differences

Solution

x	f	$f_1[\,]$		
0.6	**0.36**			
		$f[x_0, x_1] = \dfrac{3.0 - 0.36}{1.0 - 0.6} = 6.6$		
1.0	**3.00**		$\dfrac{13.8 - 6.6}{1.2 - 0.6} = 12$	
		$\dfrac{5.76 - 3.0}{1.2 - 1} = 13.8$		5
1.2	**5.76**		16	
		$\dfrac{9.8 - 5.76}{1.4 - 1.2} = 20.2$		
1.4	**9.80**			

Introduction to Divided Differences

Solution

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

- A polynomial of degree 1

$$p_1(1.1) = 0.36 + 6.6*(1.1\text{-}0.6) += 3.6600$$

To find the error compute

$$f(1.1) = 5 \times 1.1^3 - 2 \times 1.1^2 = 4.2350$$

$$|f - p_1| = |4.235 - 3.66| = 0.5750.$$

Introduction to Divided Differences

Solution

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

A polynomial of degree 2

$$p_2(1.1) = 0.36 + 6.6(1.1\text{-}0.6) + 12(1.1\text{-}0.6)(1.1\text{-}1) = 4.2600$$

$$|f - p_2| = |4.235 - 4.26| = 0.0250.$$

A polynomial of degree 3

$$p_3(1.1) = 0.36 + 6.6(1.1 - 0.6) + 12(1.1 - 0.6)(1.1 - 1)$$
$$+5(1.1 - 0.6)(1.1 - 1)(1.1 - 1.2) = 4.2350$$
$$|f - p_3| = 0.0$$

Matlab Program

Write Matlab program for Example 1

ALGORITHM 3.2

Newton's Divided-Difference Formula

To obtain the divided-difference coefficients of the interpolatory polynomial P on the $(n+1)$ distinct numbers $x_0, x_1, \ldots, x_n$ for the function f:

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), f(x_1), \ldots, f(x_n)$ as $F_{0,0}, F_{1,0}, \ldots, F_{n,0}$.

OUTPUT the numbers $F_{0,0}, F_{1,1}, \ldots, F_{n,n}$ where

$$P_n(x) = F_{0,0} + \sum_{i=1}^{n} F_{i,i} \prod_{j=0}^{i-1} (x - x_j). \quad (F_{i,i} \text{ is } f[x_0, x_1, \ldots, x_i].)$$

Step 1 For $i = 1, 2, \ldots, n$

For $j = 1, 2, \ldots, i$

set $F_{i,j} = \dfrac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$. $\quad (F_{i,j} = f[x_{i-j}, \ldots, x_i].)$

Step 2 OUTPUT $(F_{0,0}, F_{1,1}, \ldots, F_{n,n})$;

STOP.

Matlab Program

Output

```
F =

    0.3600         0         0         0
    3.0000    6.6000         0         0
    5.7600   13.8000   12.0000         0
    9.8000   20.2000   16.0000    5.0000


s =

    4.2350
```

Theorem

Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a,b]$. Then a number ξ exists in (a,b) with

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}. \qquad \blacksquare$$

Proof Let

$$g(x) = f(x) - P_n(x).$$

Since $f(x_i) = P_n(x_i)$ for each $i = 0, 1, \ldots, n$, the function g has $n+1$ distinct zeros in $[a,b]$. Generalized Rolle's Theorem 1.10 implies that a number ξ in (a,b) exists with $g^{(n)}(\xi) = 0$, so

$$0 = f^{(n)}(\xi) - P_n^{(n)}(\xi).$$

Since $P_n(x)$ is a polynomial of degree n whose leading coefficient is $f[x_0, x_1, \ldots, x_n]$,

$$P_n^{(n)}(x) = n!\, f[x_0, x_1, \ldots, x_n],$$

for all values of x. As a consequence,

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}. \qquad \blacksquare\ \blacksquare\ \blacksquare$$

Equal Spacing

Definition

For a given sequence $\{p_n\}_{n=0}^{\infty}$, the **forward difference** Δp_n (read "delta p_n") is defined by

$$\Delta p_n = p_{n+1} - p_n, \quad \text{for } n \geq 0.$$

Higher powers of the operator Δ are defined recursively by

$$\Delta^k p_n = \Delta(\Delta^{k-1} p_n), \quad \text{for } k \geq 2. \qquad \blacksquare$$

Equal Spacing

Example

Constrict the forward difference table for the give data

x	f					
1.0	3					
1.2	5.76					
1.4	9.80					
1.6	15.36					

Equal Spacing

Example

Constrict the forward difference table for the give data

x	f	Δ	Δ^2	Δ^3
1.0	3			
		$\Delta f(x_0) = 6 - 3 = 3$		
1.2	6		$\Delta^2 f(x_0) = 4 - 3 = 1$	
		$\Delta f(x_1) = 10 - 6 = 4$		$\Delta^3 f(x_0) = 0$
1.4	10		$\Delta^2 f(x_1) = 5 - 4 = 1$	
		$\Delta f(x_2) = 15 - 10 = 5$		
1.6	15			

Equal Spacing

$h = x_{i+1} - x_i$, for each $i = 0, 1, \ldots, n-1$

let $x = x_0 + sh$. $\quad x - x_i = (s - i)h.$

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$
$$+ \cdots + s(s-1)\cdots(s-n+1)h^n f[x_0, x_1, \ldots, x_n]$$
$$= f[x_0] + \sum_{k=1}^{n} s(s-1)\cdots(s-k+1)h^k f[x_0, x_1, \ldots, x_k].$$

Using binomial-coefficient notation,

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!},$$

we can express $P_n(x)$ compactly as

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k! h^k f[x_0, x_i, \ldots, x_k].$$

Equal Spacing

Forward Differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right] = \frac{1}{2h^2}\Delta^2 f(x_0),$$

and, in general,

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Newton Forward-Difference Formula

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

Equal Spacing

Example 2

a) Use the Newton forward-difference formula to construct interpolating polynomials of degree three or less for the following data.

b) Approximate $f(1.5)$.

x	f
1.0	3
1.2	6
1.4	10
1.6	15

Equal Spacing

Solution

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

x	f	Δ	Δ^2	Δ^3
1.0	3			
		3		
1.2	6		1	
		4		0
1.4	10		1	
		5		
1.6	15			

$$p_3 = 3 + 3\binom{s}{1} + \binom{s}{2} = 3 + 3s + \frac{s(s-1)}{2}$$

$$= 3 + 3s + \frac{1}{2}(s^2 - s) = 3 + 3.5s + 0.5s^2$$

$$s = \frac{x - x_0}{h} = \frac{x-1}{0.2}$$

$$p_3 = 3 + 3.5\frac{x-1}{0.2} + 0.5\left(\frac{x-1}{0.2}\right)^2$$

$$p_3(1.5) = 3 + 3.5\frac{1.5-1}{0.2} + 0.5\left(\frac{1.5-1}{0.2}\right)^2$$