Solutions of Equations in One Variable

Fixed-Point Iteration II

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

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1. Functional (Fixed-Point) Iteration
2. Convergence Criteria for the Fixed-Point Method

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3. Sample Problem: $f(x) = x^3 + 4x^2 - 10 = 0$

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Now that we have established a condition for which $g(x)$ has a unique fixed point in I, there remains the problem of how to find it. The technique employed is known as fixed-point iteration.

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- If the sequence converges to p and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

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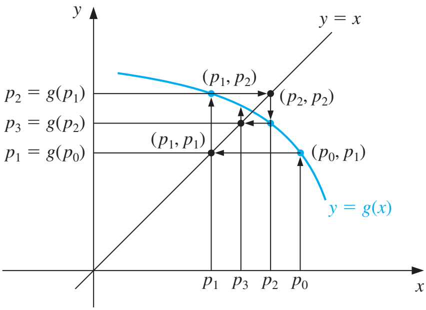
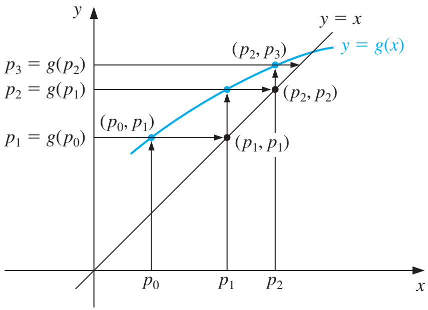
- This technique is called fixed-point, or functional iteration.

Functional (Fixed-Point) Iteration



(a)



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Functional (Fixed-Point) Iteration

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5. End of Procedure.

A Single Nonlinear Equation

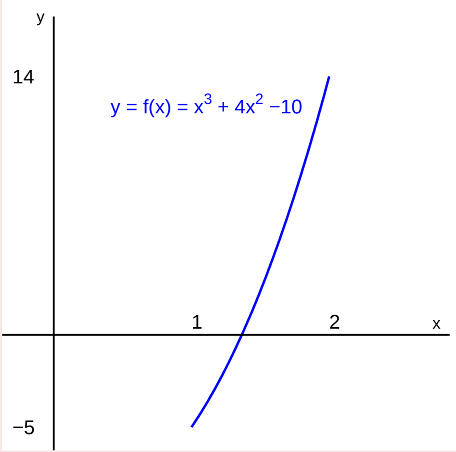
Example 1

The equation

$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Its value is approximately 1.365230013.

$f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$



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- There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

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- For example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

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- We will consider 5 such rearrangements and, later in this section, provide a brief analysis as to why some do and some not converge to $p = 1.365230013$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

5 Possible Transpositions to $x = g(x)$

$$
\begin{aligned}
x = g_1(x) &= x - x^3 - 4x^2 + 10 \\
x = g_2(x) &= \sqrt{\frac{10}{x} - 4x} \\
x = g_3(x) &= \frac{1}{2}\sqrt{10 - x^3} \\
x = g_4(x) &= \sqrt{\frac{10}{4 + x}} \\
x = g_5(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\end{aligned}
$$

Numerical Results for $f(x) = x^3 + 4x^2 - 10 = 0$

n	g_1	g_2	g_3	g_4	g_5
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$ Does not Converge

$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$ Does not Converge

$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$ Converges after 31 Iterations

$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$ Converges after 12 Iterations

$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$ Converges after 5 Iterations

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Functional (Fixed-Point) Iteration

A Crucial Question

- How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

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- How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?
- The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more importantly, some we should reject.

Functional (Fixed-Point) Iteration

Convergence Result

Let $g \in C[a, b]$ with $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $g'(x)$ exist on (a, b) with

$$|g'(x)| \leq k < 1, \qquad \forall\, x \in [a, b].$$

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$$|g'(x)| \leq k < 1, \qquad \forall\, x \in [a, b].$$

If p_0 is any point in $[a, b]$ then the sequence defined by

$$p_n = g(p_{n-1}), \qquad n \geq 1,$$

will converge to the unique fixed point p in $[a, b]$.

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- Since g maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$ and $p_n \in [a, b]$ for all n.

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- Using the Mean Value Theorem ▸ MVT and the assumption that $|g'(x)| \leq k < 1, \forall\ x \in [a, b]$, we write

$$|p_n - p| \quad = \quad |g(p_{n-1}) - g(p)|$$

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where $\xi \in (a, b)$.

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Since $k < 1$,

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n\,|p_0 - p| = 0,$$

and $\{p_n\}_{n=0}^{\infty}$ converges to p.

Functional (Fixed-Point) Iteration

Corrollary to the Convergence Result

If g satisfies the hypothesis of the Theorem, then

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|.$$

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For $n \geq 1$, the procedure used in the proof of the theorem implies that

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Functional (Fixed-Point) Iteration

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Thus, for $m > n \geq 1$,

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Functional (Fixed-Point) Iteration

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Functional (Fixed-Point) Iteration

Example: $g(x) = g(x) = 3^{-x}$

Consider the iteration function $g(x) = 3^{-x}$ over the interval $[\frac{1}{3}, 1]$ starting with $p_0 = \frac{1}{3}$. Determine a lower bound for the number of iterations n required so that $|p_n - p| < 10^{-5}$?

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Determine the Parameters of the Problem

Note that $p_1 = g(p_0) = 3^{-\frac{1}{3}} = 0.6933612$ and, since $g'(x) = -3^{-x} \ln 3$, we obtain the bound

$$|g'(x)| \;\leq\; 3^{-\frac{1}{3}} \ln 3 \;\leq\; .7617362 \;\approx\; .762 \;=\; k.$$

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We require that

$$1.513 \times 0.762^n < 10^{-5} \qquad \text{or} \qquad n > 43.88$$

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- The reason, in this case, is that we used

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whereas

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- If one had used $k = 0.602$ (were it available) to compute the bound, one would obtain $N = 23$ which is a more accurate estimate.

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A Single Nonlinear Equation

Example 2

We return to Example 1 and the equation

$$x^3 + 4x^2 - 10 = 0$$

which has a unique root in $[1, 2]$. Its value is approximately 1.365230013.

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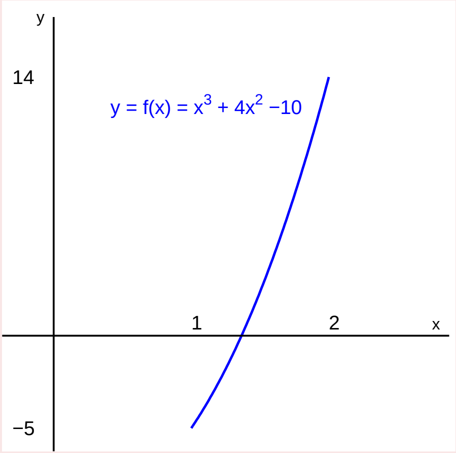

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

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x = g_1(x) &= x - x^3 - 4x^2 + 10 \\
x = g_2(x) &= \sqrt{\frac{10}{x} - 4x} \\
x = g_3(x) &= \frac{1}{2}\sqrt{10 - x^3} \\
x = g_4(x) &= \sqrt{\frac{10}{4 + x}} \\
x = g_5(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\end{aligned}
$$

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

Results Observed for $x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$	Does not Converge
$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$	Converges after 31 Iterations
$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$	Converges after 12 Iterations
$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$	Converges after 5 Iterations

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\dfrac{10}{x} - 4x}$	Does not Converge
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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

Iteration for $x = g_1(x)$ Does Not Converge

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

Iteration for $x = g_1(x)$ Does Not Converge

Since

$$g_1'(x) = 1 - 3x^2 - 8x \qquad g_1'(1) = -10 \qquad g_1'(2) = -27$$

there is no interval $[a, b]$ containing p for which $\left|g_1'(x)\right| < 1$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

Iteration for $x = g_1(x)$ Does Not Converge

Since

$$g_1'(x) = 1 - 3x^2 - 8x \qquad g_1'(1) = -10 \qquad g_1'(2) = -27$$

there is no interval $[a, b]$ containing p for which $\left|g_1'(x)\right| < 1$. Also, note that

$$g_1(1) = 6 \qquad \text{and} \qquad g_2(2) = -12$$

so that $g(x) \notin [1, 2]$ for $x \in [1, 2]$.

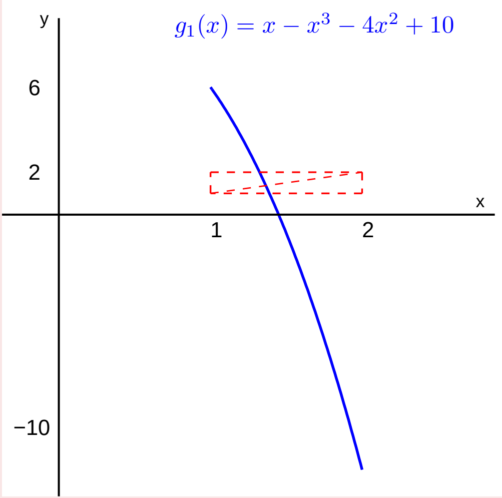
Iteration Function: $x = g_1(x) = x - x^3 - 4x^2 + 10$

Iterations starting with $p_0 = 1.5$

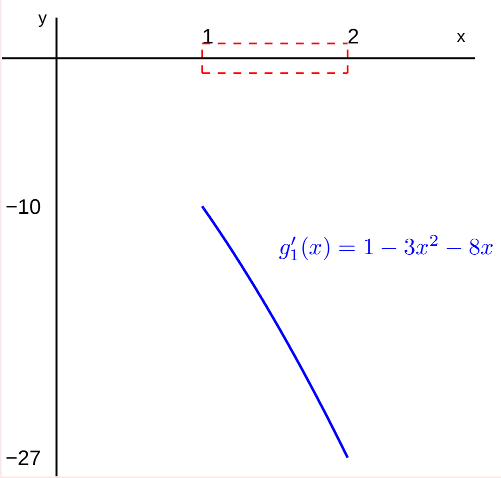
n	p_{n-1}	p_n	$\lvert p_n - p_{n-1} \rvert$
1	1.5000000	-0.8750000	2.3750000
2	-0.8750000	6.7324219	7.6074219
3	6.7324219	-469.7200120	476.4524339

$$p_4 \approx 1.03 \times 10^8$$

g_1 Does Not Map $[1, 2]$ into $[1, 2]$



$|g_1'(x)| > 1$ on $[1, 2]$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\dfrac{10}{x} - 4x}$	Does not Converge
$x = g_3(x) = \dfrac{1}{2}\sqrt{10 - x^3}$	Converges after 31 Iterations
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$x = g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$	Converges after 5 Iterations

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

Iteration for $x = g_2(x)$ is Not Defined

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

Iteration for $x = g_2(x)$ is Not Defined

It is clear that $g_2(x)$ does not map $[1, 2]$ onto $[1, 2]$ and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined for $p_0 = 1.5$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

Iteration for $x = g_2(x)$ is Not Defined

It is clear that $g_2(x)$ does not map $[1, 2]$ onto $[1, 2]$ and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined for $p_0 = 1.5$. Also, there is no interval containing p such that

$$\left|g_2'(x)\right| < 1$$

since

$$g'(1) \approx -2.86 \qquad\qquad g'(p) \approx -3.43$$

and $g'(x)$ is not defined for $x > 1.58$.

Iteration Function: $x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$

Iterations starting with $p_0 = 1.5$

n	p_{n-1}	p_n	$\lvert p_n - p_{n-1} \rvert$
1	1.5000000	0.8164966	0.6835034
2	0.8164966	2.9969088	2.1804122
3	2.9969088	$\sqrt{-8.6509}$	—

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$	Does not Converge
$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$	Converges after 31 Iterations
$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$	Converges after 12 Iterations
$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$	Converges after 5 Iterations

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

Iteration for $x = g_3(x)$ Converges (Slowly)

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

Iteration for $x = g_3(x)$ Converges (Slowly)

By differentiation,

$$g_3'(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \qquad \text{for} \quad x \in [1, 2]$$

and so g=g_3 is strictly decreasing on $[1, 2]$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

Iteration for $x = g_3(x)$ Converges (Slowly)

By differentiation,

$$g_3'(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \qquad \text{for} \quad x \in [1, 2]$$

and so g=g_3 is strictly decreasing on $[1, 2]$. However, $\left|g_3'(x)\right| > 1$ for $x > 1.71$ and $\left|g_3'(2)\right| \approx -2.12$.

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

Iteration for $x = g_3(x)$ Converges (Slowly)

By differentiation,

$$g_3'(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \qquad \text{for} \quad x \in [1, 2]$$

and so g=g_3 is strictly decreasing on $[1, 2]$. However, $\left|g_3'(x)\right| > 1$ for $x > 1.71$ and $\left|g_3'(2)\right| \approx -2.12$. A closer examination of $\{p_n\}_{n=0}^{\infty}$ will show that it suffices to consider the interval $[1, 1.7]$ where $\left|g_3'(x)\right| < 1$ and $g(x) \in [1, 1.7]$ for $x \in [1, 1.7]$.

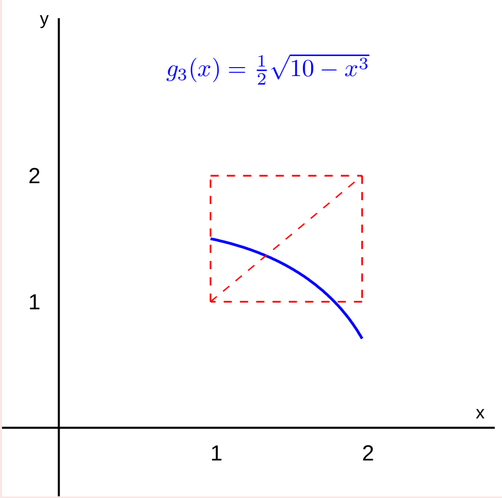
Iteration Function: $x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$

Iterations starting with $p_0 = 1.5$

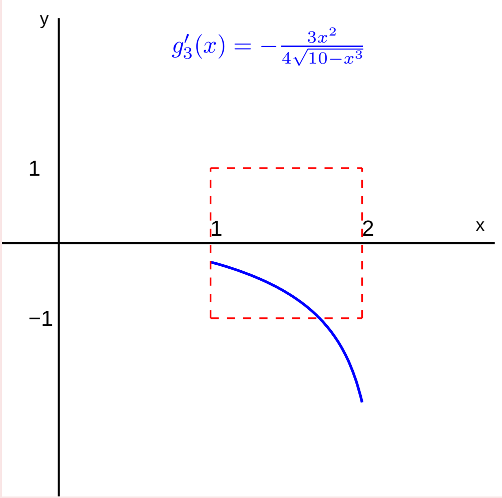
n	p_{n-1}	p_n	$\|p_n - p_{n-1}\|$
1	1.500000000	1.286953768	0.213046232
2	1.286953768	1.402540804	0.115587036
3	1.402540804	1.345458374	0.057082430
4	1.345458374	1.375170253	0.029711879
5	1.375170253	1.360094193	0.015076060
6	1.360094193	1.367846968	0.007752775

30	1.365230013	1.365230014	0.000000001
31	1.365230014	1.365230013	0.000000000

g_3 Maps $[1, 1.7]$ into $[1, 1.7]$



$|g_3'(x)| < 1$ on $[1, 1.7]$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$	Does not Converge
$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$	Converges after 31 Iterations
$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$	Converges after 12 Iterations
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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}}$$

Iteration for $x = g_4(x)$ Converges (Moderately)

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}}$$

Iteration for $x = g_4(x)$ Converges (Moderately)

By differentiation,

$$g_4'(x) = -\sqrt{\frac{10}{4(4+x)^3}} < 0$$

and it is easy to show that

$$0.10 < \left|g_4'(x)\right| < 0.15 \qquad \forall\ x \in [1, 2]$$

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_4(x) = \sqrt{\frac{10}{4+x}}$$

Iteration for $x = g_4(x)$ Converges (Moderately)

By differentiation,

$$g_4'(x) = -\sqrt{\frac{10}{4(4+x)^3}} < 0$$

and it is easy to show that

$$0.10 < \left|g_4'(x)\right| < 0.15 \qquad \forall\ x \in [1, 2]$$

The bound on the magnitude of $\left|g_4'(x)\right|$ is much smaller than that for $\left|g_3'(x)\right|$ and this explains the reason for the much faster convergence.

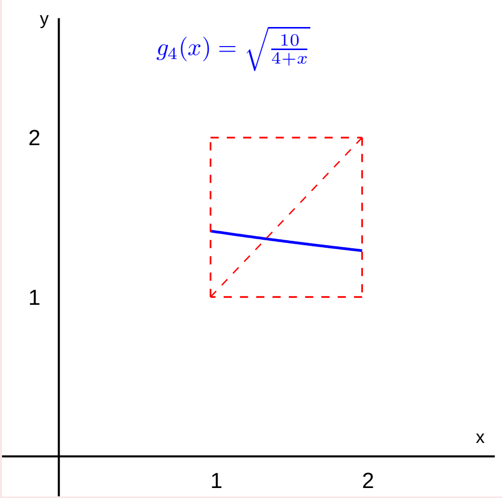
Iteration Function: $x = g_4(x) = \sqrt{\frac{10}{4+x}}$

Iterations starting with $p_0 = 1.5$

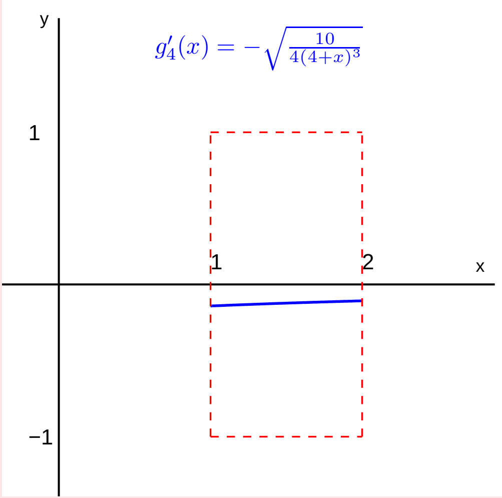
n	p_{n-1}	p_n	$\lvert p_n - p_{n-1} \rvert$
1	1.500000000	1.348399725	0.151600275
2	1.348399725	1.367376372	0.018976647
3	1.367376372	1.364957015	0.002419357
4	1.364957015	1.365264748	0.000307733
5	1.365264748	1.365225594	0.000039154
6	1.365225594	1.365230576	0.000004982

11	1.365230014	1.365230013	0.000000000
12	1.365230013	1.365230013	0.000000000

g_4 Maps $[1, 2]$ into $[1, 2]$



$|g_4'(x)| < 1$ on $[1, 2]$



Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
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Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Iteration for $x = g_5(x)$ Converges (Rapidly)

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Iteration for $x = g_5(x)$ Converges (Rapidly)

For the iteration function $g_5(x)$, we obtain:

$$g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g_5'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow g_5'(p) = 0$$

Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

Iteration for $x = g_5(x)$ Converges (Rapidly)

For the iteration function $g_5(x)$, we obtain:

$$g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g_5'(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow g_5'(p) = 0$$

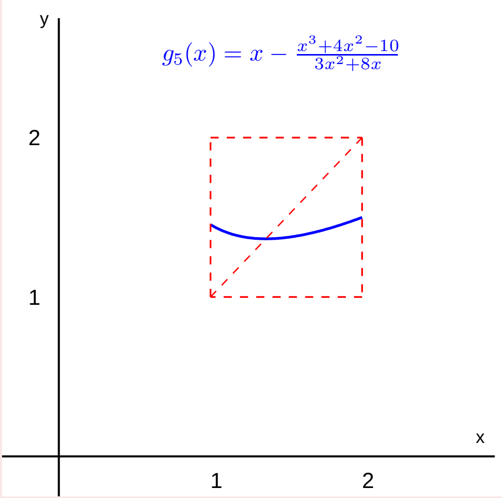
It is straightforward to show that $0 \leq \left|g_5'(x)\right| < 0.28 \;\; \forall \, x \in [1, 2]$ and the order of convergence is quadratic since $g_5'(p) = 0$.

Iteration Function: $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

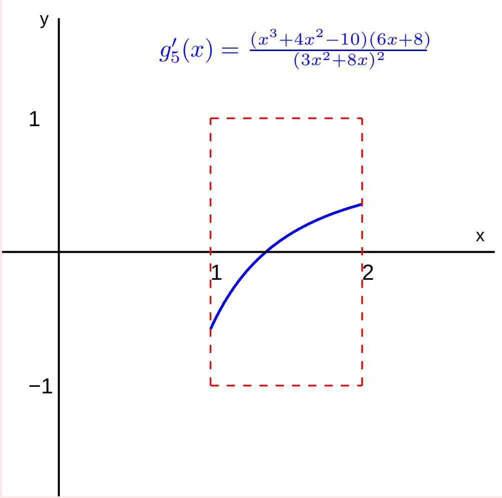
Iterations starting with $p_0 = 1.5$

n	p_{n-1}	p_n	$\lvert p_n - p_{n-1}\rvert$
1	1.500000000	1.373333333	0.126666667
2	1.373333333	1.365262015	0.008071318
3	1.365262015	1.365230014	0.000032001
4	1.365230014	1.365230013	0.000000001
5	1.365230013	1.365230013	0.000000000

g_5 Maps $[1,2]$ into $[1,2]$



$|g_5'(x)| < 1$ on $[1, 2]$



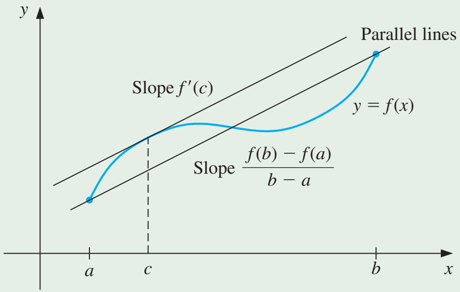
Questions?

Reference Material

Mean Value Theorem

If $f \in C[a, b]$ and f is differentiable on (a, b), then a number c exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Return to Fixed-Point Convergence Theorem