## 5.4 INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM

We saw in Section 5.3 that the second part of the Fundamental Theorem of Calculus provides a very powerful method for evaluating the definite integral of a function, assuming that we can find an antiderivative of the function. In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals. We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

### INDEFINITE INTEGRALS

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if $f$ is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of $f$. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where $F$ is an antiderivative of $f$.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of $f$ and is called an **indefinite integral**. Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire family of functions (one antiderivative for each value of the constant $C$).

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function (or family of functions). The connection between them is given by Part 2 of the Fundamental Theorem. If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions. We therefore restate the Table of Antidifferentiation Formulas from Section 4.9, together with a few others, in the notation of indefinite integrals. Any formula can be verified by differentiating the function on the right side and obtaining the integrand. For instance

$$\int \sec^2 x\,dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

**1 TABLE OF INDEFINITE INTEGRALS**

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Recall from Theorem 4.9.1 that the most general antiderivative on a given interval is obtained by adding a constant to a particular antiderivative. **We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.** Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$. This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

■ The indefinite integral in Example 1 is graphed in Figure 1 for several values of $C$. The value of $C$ is the $y$-intercept.

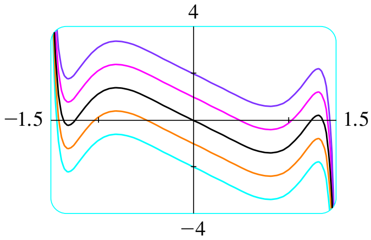



**FIGURE 1**

**EXAMPLE 1** Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

You should check this answer by differentiating it. ■

**EXAMPLE 2** Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

SOLUTION This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta \cot\theta\,d\theta = -\csc\theta + C$$

**EXAMPLE 3** Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

SOLUTION Using FTC2 and Table 1, we have

$$\int_0^3 (x^3 - 6x)\,dx = \frac{x^4}{4} - 6\frac{x^2}{2}\bigg]_0^3$$

$$= \left(\tfrac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\tfrac{1}{4}\cdot 0^4 - 3\cdot 0^2\right)$$

$$= \tfrac{81}{4} - 27 - 0 + 0 = -6.75$$

Compare this calculation with Example 2(b) in Section 5.2.

**EXAMPLE 4** Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx$ and interpret the result in terms of areas.

■ Figure 2 shows the graph of the integrand in Example 4. We know from Section 5.2 that the value of the integral can be interpreted as the sum of the areas labeled with a plus sign minus the area labeled with a minus sign.

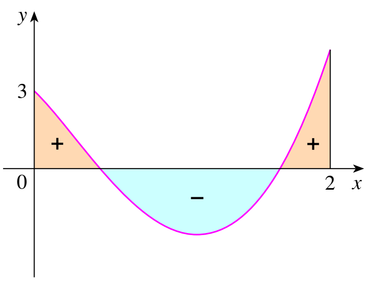



**FIGURE 2**

SOLUTION The Fundamental Theorem gives

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx = 2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\bigg]_0^2$$

$$= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2$$

$$= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0$$

$$= -4 + 3\tan^{-1}2$$

This is the exact value of the integral. If a decimal approximation is desired, we can use a calculator to approximate $\tan^{-1}2$. Doing so, we get

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx \approx -0.67855$$

**EXAMPLE 5** Evaluate $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\bigg]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\bigg]_1^9$$

$$= \left(2\cdot 9 + \tfrac{2}{3}\cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2\cdot 1 + \tfrac{2}{3}\cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$

## APPLICATIONS

Part 2 of the Fundamental Theorem says that if $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to $x$ and $F(b) - F(a)$ is the change in $y$ when $x$ changes from $a$ to $b$. [Note that $y$ could, for instance, increase, then decrease, then increase again. Although $y$ might change in both directions, $F(b) - F(a)$ represents the net change in $y$.] So we can reformulate FTC2 in words as follows.

**THE NET CHANGE THEOREM** The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences that we discussed in Section 3.7. Here are a few instances of this idea:

- If $V(t)$ is the volume of water in a reservoir at time $t$, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time $t$. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

  is the change in the amount of water in the reservoir between time $t_1$ and time $t_2$.

- If $[C](t)$ is the concentration of the product of a chemical reaction at time $t$, then the rate of reaction is the derivative $d[C]/dt$. So

$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$$

  is the change in the concentration of C from time $t_1$ to time $t_2$.

- If the mass of a rod measured from the left end to a point $x$ is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

  is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

- If the rate of growth of a population is $dn/dt$, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

  is the net change in population during the time period from $t_1$ to $t_2$. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If $C(x)$ is the cost of producing $x$ units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from $x_1$ units to $x_2$ units.

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

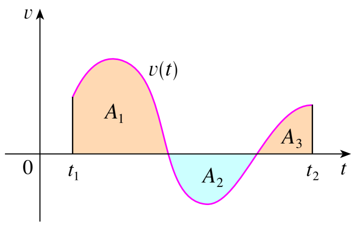
**2** $$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or displacement, of the particle during the time period from $t_1$ to $t_2$. In Section 5.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during that time interval, we have to consider the intervals when $v(t) \geqslant 0$ (the particle moves to the right) and also the intervals when $v(t) \leqslant 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

**3** $$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled}$$

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.




$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

**FIGURE 3**

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time $t_1$ to time $t_2$.

**V EXAMPLE 6** A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 - t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \leqslant t \leqslant 4$.
(b) Find the distance traveled during this time period.

SOLUTION
(a) By Equation 2, the displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$
$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.

(b) Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leqslant 0$ on the interval $[1, 3]$ and $v(t) \geqslant 0$ on $[3, 4]$. Thus, from Equation 3, the distance traveled is

■ To integrate the absolute value of $v(t)$, we use Property 5 of integrals from Section 5.2 to split the integral into two parts, one where $v(t) \leqslant 0$ and one where $v(t) \geqslant 0$.

$$\begin{aligned}\int_1^4 |v(t)|\,dt &= \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \\ &= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\ &= \frac{61}{6} \approx 10.17 \text{ m}\end{aligned}$$

■

**EXAMPLE 7** Figure 4 shows the power consumption in the city of San Francisco for a day in September ($P$ is measured in megawatts; $t$ is measured in hours starting at midnight). Estimate the energy used on that day.

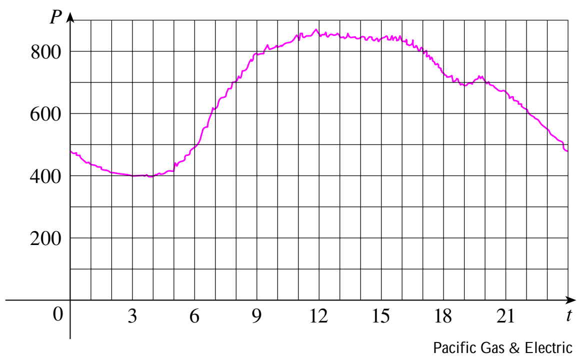



**FIGURE 4**

SOLUTION Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$:

$$\begin{aligned}\int_0^{24} P(t)\,dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\,\Delta t \\ &\approx (440 + 400 + 420 + 620 + 790 + 840 + 850 \\ &\qquad + 840 + 810 + 690 + 670 + 550)(2) \\ &= 15{,}840\end{aligned}$$

The energy used was approximately 15,840 megawatt-hours. ■

■ A note on units

How did we know what units to use for energy in Example 7? The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$. Now $P(t_i^*)$ is measured in megawatts and $\Delta t$ is measured in hours, so their product is measured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for $x$.

## 5.4 EXERCISES

**1–4** Verify by differentiation that the formula is correct.

1. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$

2. $\displaystyle\int x\cos x\,dx = x\sin x + \cos x + C$

3. $\displaystyle\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x + C$

4. $\displaystyle\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$

**5–18** Find the general indefinite integral.

5. $\displaystyle\int (x^2 + x^{-2})\,dx$

6. $\displaystyle\int \left(\sqrt{x^3} + \sqrt[3]{x^2}\right)dx$

7. $\displaystyle\int \left(x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x - 2\right)dx$

8. $\displaystyle\int (y^3 + 1.8y^2 - 2.4y)\,dy$

9. $\displaystyle\int (1-t)(2+t^2)\,dt$

10. $\displaystyle\int v(v^2+2)^2\,dv$

11. $\displaystyle\int \frac{x^3 - 2\sqrt{x}}{x}\,dx$

12. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2+1}\right)dx$

13. $\displaystyle\int (\sin x + \sinh x)\,dx$

14. $\displaystyle\int (\csc^2 t - 2e^t)\,dt$

15. $\displaystyle\int (\theta - \csc\theta\cot\theta)\,d\theta$

16. $\displaystyle\int \sec t(\sec t + \tan t)\,dt$

17. $\displaystyle\int (1 + \tan^2\alpha)\,d\alpha$

18. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx$

**19–20** Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

19. $\displaystyle\int \left(\cos x + \tfrac{1}{2}x\right)dx$

20. $\displaystyle\int (e^x - 2x^2)\,dx$

**21–44** Evaluate the integral.

21. $\displaystyle\int_0^2 (6x^2 - 4x + 5)\,dx$

22. $\displaystyle\int_1^3 (1 + 2x - 4x^3)\,dx$

23. $\displaystyle\int_{-1}^0 (2x - e^x)\,dx$

24. $\displaystyle\int_{-2}^0 (u^5 - u^3 + u^2)\,du$

25. $\displaystyle\int_{-2}^2 (3u+1)^2\,du$

26. $\displaystyle\int_0^4 (2v+5)(3v-1)\,dv$

27. $\displaystyle\int_1^4 \sqrt{t}\,(1+t)\,dt$

28. $\displaystyle\int_0^9 \sqrt{2t}\,dt$

29. $\displaystyle\int_{-2}^{-1} \left(4y^3 + \frac{2}{y^3}\right)dy$

30. $\displaystyle\int_1^2 \frac{y + 5y^7}{y^3}\,dy$

31. $\displaystyle\int_0^1 x\left(\sqrt[3]{x} + \sqrt[4]{x}\right)dx$

32. $\displaystyle\int_0^5 (2e^x + 4\cos x)\,dx$

33. $\displaystyle\int_1^4 \sqrt{\frac{5}{x}}\,dx$

34. $\displaystyle\int_1^9 \frac{3x-2}{\sqrt{x}}\,dx$

35. $\displaystyle\int_0^{\pi} (4\sin\theta - 3\cos\theta)\,d\theta$

36. $\displaystyle\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta\,d\theta$

37. $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$

38. $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

39. $\displaystyle\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx$

40. $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx$

41. $\displaystyle\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt$

42. $\displaystyle\int_1^2 \frac{(x-1)^3}{x^2}\,dx$

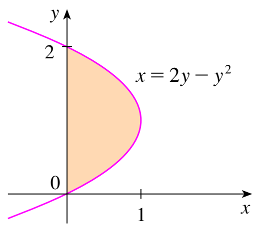
43. $\displaystyle\int_{-1}^2 (x - 2|x|)\,dx$

44. $\displaystyle\int_0^{3\pi/2} |\sin x|\,dx$

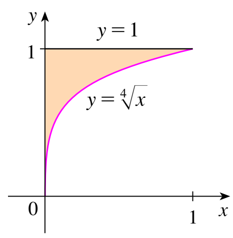
45. Use a graph to estimate the $x$-intercepts of the curve $y = x + x^2 - x^4$. Then use this information to estimate the area of the region that lies under the curve and above the $x$-axis.

46. Repeat Exercise 45 for the curve $y = 2x + 3x^4 - 2x^6$.

47. The area of the region that lies to the right of the $y$-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\,dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.




48. The boundaries of the shaded region are the $y$-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing $x$ as a function of $y$ and integrating with respect to $y$ (as in Exercise 47).



**49.** If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

**50.** The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 3 in Section 3.7.) What does $\int_a^b I(t)\,dt$ represent?

**51.** If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time $t$, what does $\int_0^{120} r(t)\,dt$ represent?

**52.** A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?

**53.** In Section 4.7 we defined the marginal revenue function $R'(x)$ as the derivative of the revenue function $R(x)$, where $x$ is the number of units sold. What does $\int_{1000}^{5000} R'(x)\,dx$ represent?

**54.** If $f(x)$ is the slope of a trail at a distance of $x$ miles from the start of the trail, what does $\int_3^5 f(x)\,dx$ represent?

**55.** If $x$ is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\,dx$?

**56.** If the units for $x$ are feet and the units for $a(x)$ are pounds per foot, what are the units for $da/dx$? What units does $\int_2^8 a(x)\,dx$ have?

**57–58** The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

**57.** $v(t) = 3t - 5, \quad 0 \leqslant t \leqslant 3$

**58.** $v(t) = t^2 - 2t - 8, \quad 1 \leqslant t \leqslant 6$

**59–60** The acceleration function (in $m/s^2$) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time $t$ and (b) the distance traveled during the given time interval.

**59.** $a(t) = t + 4, \quad v(0) = 5, \quad 0 \leqslant t \leqslant 10$

**60.** $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \leqslant t \leqslant 3$

**61.** The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where $x$ is measured in meters from one end of the rod. Find the total mass of the rod.

**62.** Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leqslant t \leqslant 50$. Find the amount of water that flows from the tank during the first 10 minutes.

**63.** The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

| $t$ (s) | $v$ (mi/h) | $t$ (s) | $v$ (mi/h) |
|---|---|---|---|
| 0 | 0 | 60 | 56 |
| 10 | 38 | 70 | 53 |
| 20 | 52 | 80 | 50 |
| 30 | 58 | 90 | 47 |
| 40 | 55 | 100 | 45 |
| 50 | 51 | | |

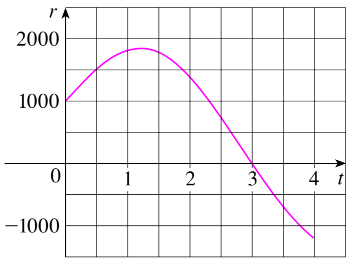
**64.** Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time $t$ is measured in seconds and the units for $r(t)$ are tonnes (metric tons) per second.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $r(t)$ | 2 | 10 | 24 | 36 | 46 | 54 | 60 |

(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after 6 seconds.
(b) Use the Midpoint Rule to estimate $Q(6)$.

**65.** The marginal cost of manufacturing $x$ yards of a certain fabric is $C'(x) = 3 - 0.01x + 0.000006x^2$ (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

**66.** Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown. If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule to estimate the amount of water four days later.




**67.** Economists use a cumulative distribution called a Lorenz curve to describe the distribution of income between households in a given country. Typically, a Lorenz curve is defined on $[0, 1]$ with endpoints $(0, 0)$ and $(1, 1)$, and is continuous, increasing, and concave upward. The points on this curve are determined by ranking all households by income and then computing the percentage of households whose income is less than or equal to a given percentage of the total income of the country. For example, the point $(a/100, b/100)$ is on the Lorenz curve if the bottom $a$% of the households receive less than or equal to $b$% of the total income. Absolute equality of income distribution would occur if the bottom $a$% of the