2. Carl Boyer, The History of the Calculus and Its Conceptual Development (New York: Dover, 1959), Chapter V.
3. C. H. Edwards, The Historical Development of the Calculus (New York: Springer-Verlag, 1979), Chapters 8 and 9.
4. Howard Eves, An Introduction to the History of Mathematics, 6th ed. (New York: Saunders, 1990), Chapter 11.
5. C. C. Gillispie, ed., Dictionary of Scientific Biography (New York: Scribner's, 1974). See the article on Leibniz by Joseph Hofmann in Volume VIII and the article on Newton by I. B. Cohen in Volume X.
6. Victor Katz, A History of Mathematics: An Introduction (New York: HarperCollins, 1993), Chapter 12.
7. Morris Kline, Mathematical Thought from Ancient to Modern Times (New York: Oxford University Press, 1972), Chapter 17.

Sourcebooks

1. John Fauvel and Jeremy Gray, eds., The History of Mathematics: A Reader (London: MacMillan Press, 1987), Chapters 12 and 13.
2. D. E. Smith, ed., A Sourcebook in Mathematics (New York: Dover, 1959), Chapter V.
3. D. J. Struik, ed., A Sourcebook in Mathematics, 1200–1800 (Princeton, N.J.: Princeton University Press, 1969), Chapter V.

5.5 THE SUBSTITUTION RULE

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\int 2x\sqrt{1+x^2}\,dx \tag{1}$$

To find this integral we use the problem-solving strategy of introducing something extra. Here the "something extra" is a new variable; we change from the variable x to a new variable u. Suppose that we let u be the quantity under the root sign in (1), $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$. Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in (1) and so, formally, without justifying our calculation, we could write

■ Differentials were defined in Section 3.10. If $u = f(x)$, then

$$du = f'(x)\,dx$$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du \tag{2}$$
$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2+1)^{3/2} + C$$

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2+1)^{3/2} + C\right] = \tfrac{2}{3}\cdot\tfrac{3}{2}(x^2+1)^{1/2}\cdot 2x = 2x\sqrt{x^2+1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))g'(x)\,dx$. Observe that if $F' = f$, then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \tag{3}$$

because, by the Chain Rule,

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

If we make the "change of variable" or "substitution" $u = g(x)$, then from Equation 3 we have

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du$$

or, writing $F' = f$, we get

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Thus we have proved the following rule.

4 THE SUBSTITUTION RULE If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation. Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in (4) as differentials.

Thus the Substitution Rule says: **It is permissible to operate with dx and du after integral signs as if they were differentials.**

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\,dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\,dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\,dx = du/4$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\,dx = \int \cos u \cdot \tfrac{1}{4}\,du = \tfrac{1}{4}\int \cos u\,du$$

$$= \tfrac{1}{4}\sin u + C$$

$$= \tfrac{1}{4}\sin(x^4 + 2) + C$$

■ Check the answer by differentiating it.

Notice that at the final stage we had to return to the original variable x. ■

The idea behind the Substitution Rule is to replace a relatively complicated integral by a simpler integral. This is accomplished by changing from the original variable x to a new variable u that is a function of x. Thus, in Example 1, we replaced the integral $\int x^3 \cos(x^4 + 2)\,dx$ by the simpler integral $\frac{1}{4}\int \cos u\,du$.

The main challenge in using the Substitution Rule is to think of an appropriate substitution. You should try to choose u to be some function in the integrand whose differential also occurs (except for a constant factor). This was the case in Example 1. If that is not possible, try choosing u to be some complicated part of the integrand (perhaps the inner function in a composite function). Finding the right substitution is a bit of an art. It's not unusual to guess wrong; if your first guess doesn't work, try another substitution.

EXAMPLE 2 Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = du/2$. Thus the Substitution Rule gives

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2} = \tfrac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$
$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x+1}$. Then

$$du = \frac{dx}{\sqrt{2x+1}} \qquad \text{so} \qquad dx = \sqrt{2x+1}\,du = u\,du$$

(Or observe that $u^2 = 2x + 1$, so $2u\,du = 2\,dx$.) Therefore

$$\int \sqrt{2x+1}\,dx = \int u \cdot u\,du = \int u^2\,du$$
$$= \frac{u^3}{3} + C = \tfrac{1}{3}(2x+1)^{3/2} + C$$

V EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\tfrac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\tfrac{1}{8}\int u^{-1/2}\,du$$
$$= -\tfrac{1}{8}\left(2\sqrt{u}\right) + C = -\tfrac{1}{4}\sqrt{1-4x^2} + C$$

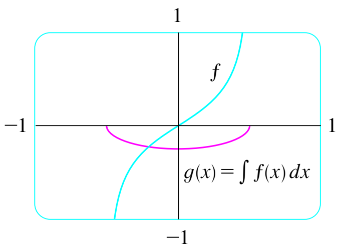


FIGURE 1

$f(x) = \dfrac{x}{\sqrt{1-4x^2}}$

$g(x) = \int f(x)\,dx = -\tfrac{1}{4}\sqrt{1-4x^2}$

The answer to Example 3 could be checked by differentiation, but instead let's check it with a graph. In Figure 1 we have used a computer to graph both the integrand $f(x) = x/\sqrt{1-4x^2}$ and its indefinite integral $g(x) = -\frac{1}{4}\sqrt{1-4x^2}$ (we take the case $C = 0$). Notice that $g(x)$ decreases when $f(x)$ is negative, increases when $f(x)$ is positive, and has its minimum value when $f(x) = 0$. So it seems reasonable, from the graphical evidence, that g is an antiderivative of f.

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \tfrac{1}{5}\int e^u\,du = \tfrac{1}{5}e^u + C = \tfrac{1}{5}e^{5x} + C$$

EXAMPLE 5 Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\, dx$, so $x\, dx = du/2$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned}
\int \sqrt{1 + x^2}\, x^5\, dx &= \int \sqrt{1 + x^2}\, x^4 \cdot x\, dx \\
&= \int \sqrt{u}\, (u - 1)^2 \frac{du}{2} = \tfrac{1}{2} \int \sqrt{u}\, (u^2 - 2u + 1)\, du \\
&= \tfrac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du \\
&= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2 \cdot \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\
&= \tfrac{1}{7}(1 + x^2)^{7/2} - \tfrac{2}{5}(1 + x^2)^{5/2} + \tfrac{1}{3}(1 + x^2)^{3/2} + C
\end{aligned}$$

■

V **EXAMPLE 6** Calculate $\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C
\end{aligned}$$

■

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 6 can also be written as

5 $$\int \tan x\, dx = \ln|\sec x| + C$$

DEFINITE INTEGRALS

When evaluating a definite integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem. For instance, using the result of Example 2, we have

$$\begin{aligned}
\int_0^4 \sqrt{2x + 1}\, dx &= \int \sqrt{2x + 1}\, dx\Big]_0^4 = \tfrac{1}{3}(2x + 1)^{3/2}\Big]_0^4 \\
&= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27 - 1) = \tfrac{26}{3}
\end{aligned}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

■ This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

6 THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Let F be an antiderivative of f. Then, by (3), $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

EXAMPLE 7 Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = du/2$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\,du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

■ The geometric interpretation of Example 7 is shown in Figure 2. The substitution $u = 2x + 1$ stretches the interval $[0, 4]$ by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

Observe that when using (6) we do not return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u. ■

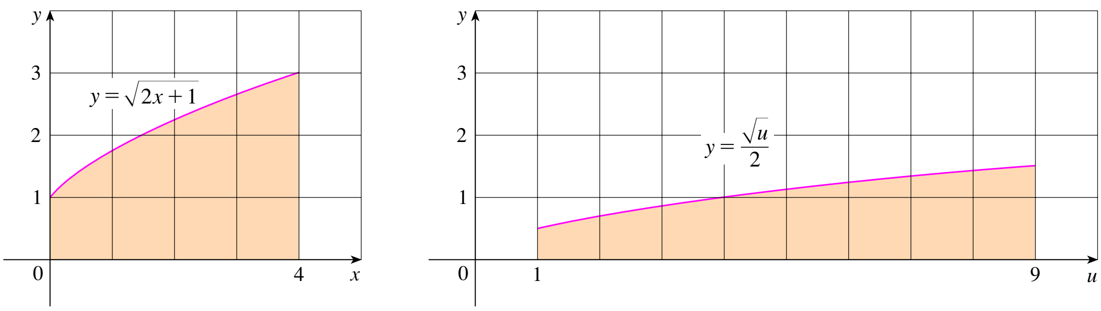


FIGURE 2

■ The integral given in Example 8 is an abbreviation for

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx$$

EXAMPLE 8 Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

SOLUTION Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -du/5$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\int_1^2 \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2}$$
$$= -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Big]_{-2}^{-7}$$
$$= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}$$

EXAMPLE 9 Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$

■ Since the function $f(x) = (\ln x)/x$ in Example 9 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 3.

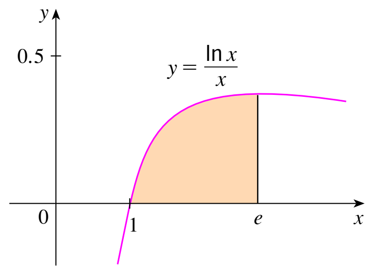


FIGURE 3

SYMMETRY

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

7 INTEGRALS OF SYMMETRIC FUNCTIONS Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF We split the integral in two:

8 $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_0^{-a} f(x)\,dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)\,du$$

and so Equation 8 becomes

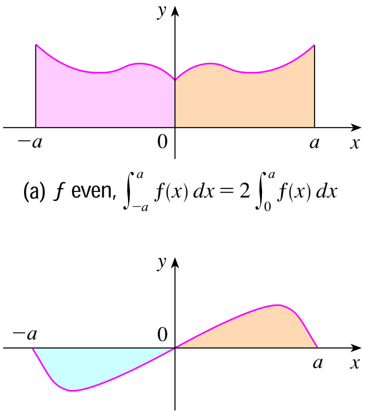
$$\boxed{9} \qquad \int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

(a) If f is even, then $f(-u) = f(u)$ so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

FIGURE 4

Theorem 7 is illustrated by Figure 4. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. Thus part (b) says the integral is 0 because the areas cancel.

EXAMPLE 10 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_0^2 (x^6 + 1)\,dx$$

$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7}$$

EXAMPLE 11 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

5.5 EXERCISES

1–6 Evaluate the integral by making the given substitution.

1. $\int e^{-x}\,dx, \quad u = -x$

2. $\int x^3(2 + x^4)^5\,dx, \quad u = 2 + x^4$

3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\int \frac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$

5. $\int \cos^3\theta \sin\theta\,d\theta, \quad u = \cos\theta$

6. $\int \frac{\sec^2(1/x)}{x^2}\,dx, \quad u = 1/x$

7–46 Evaluate the indefinite integral.

7. $\int x \sin(x^2)\,dx$

8. $\int x^2(x^3 + 5)^9\,dx$

9. $\int (3x - 2)^{20}\,dx$

10. $\int (3t + 2)^{2.4}\,dt$

11. $\int (x + 1)\sqrt{2x + x^2}\,dx$

12. $\int \frac{x}{(x^2 + 1)^2}\,dx$

13. $\int \frac{dx}{5 - 3x}$

14. $\int e^x \sin(e^x)\,dx$

15. $\int \sin \pi t\,dt$

16. $\int \frac{x}{x^2 + 1}\,dx$

17. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

18. $\int \sec 2\theta \tan 2\theta\,d\theta$

19. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

20. $\displaystyle\int \frac{dx}{ax+b} \quad (a \neq 0)$

21. $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

22. $\displaystyle\int \sqrt{x}\,\sin(1+x^{3/2})\,dx$

23. $\displaystyle\int \cos\theta\,\sin^6\theta\,d\theta$

24. $\displaystyle\int (1+\tan\theta)^5\sec^2\theta\,d\theta$

25. $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

26. $\displaystyle\int e^{\cos t}\sin t\,dt$

27. $\displaystyle\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

28. $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx$

29. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

30. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

31. $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx$

32. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

33. $\displaystyle\int \sqrt{\cot x}\,\csc^2 x\,dx$

34. $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

35. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

36. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

37. $\displaystyle\int \cot x\,dx$

38. $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

39. $\displaystyle\int \sec^3 x\tan x\,dx$

40. $\displaystyle\int \sin t\,\sec^2(\cos t)\,dt$

41. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\,\sin^{-1}x}$

42. $\displaystyle\int \frac{x}{1+x^4}\,dx$

43. $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

44. $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\,dx$

45. $\displaystyle\int \frac{x}{\sqrt[4]{x+2}}\,dx$

46. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

47–50 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

47. $\displaystyle\int x(x^2-1)^3\,dx$

48. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

49. $\displaystyle\int \sin^3 x\cos x\,dx$

50. $\displaystyle\int \tan^2\theta\,\sec^2\theta\,d\theta$

51–70 Evaluate the definite integral.

51. $\displaystyle\int_0^2 (x-1)^{25}\,dx$

52. $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

53. $\displaystyle\int_0^1 x^2(1+2x^3)^5\,dx$

54. $\displaystyle\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

55. $\displaystyle\int_0^{\pi} \sec^2(t/4)\,dt$

56. $\displaystyle\int_{1/6}^{1/2} \csc\pi t\cot\pi t\,dt$

57. $\displaystyle\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta$

58. $\displaystyle\int_0^1 xe^{-x^2}\,dx$

59. $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

60. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

61. $\displaystyle\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

62. $\displaystyle\int_0^{\pi/2} \cos x\sin(\sin x)\,dx$

63. $\displaystyle\int_0^a x\sqrt{x^2+a^2}\,dx \quad (a > 0)$

64. $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$

65. $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

66. $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

67. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

68. $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

69. $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

70. $\displaystyle\int_0^{T/2} \sin(2\pi t/T-\alpha)\,dt$

71–72 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

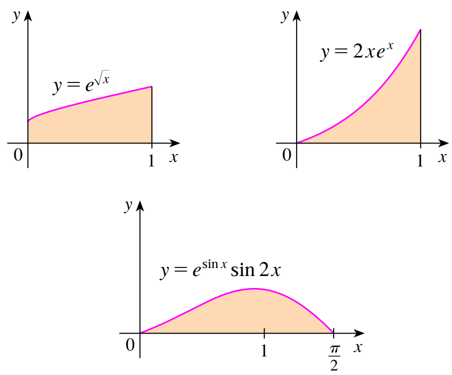
71. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

72. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

73. Evaluate $\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

74. Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

75. Which of the following areas are equal? Why?



76. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\,dt$, over a 24-hour time period?

77. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

78. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

79. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

80. Alabama Instruments Company has set up a production line to manufacture a new calculator. The rate of production of these calculators after t weeks is

$$\frac{dx}{dt} = 5000\left(1 - \frac{100}{(t+10)^2}\right) \text{ calculators/week}$$

(Notice that production approaches 5000 per week as time goes on, but the initial production is lower because of the workers' unfamiliarity with the new techniques.) Find the number of calculators produced from the beginning of the third week to the end of the fourth week.

81. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

82. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$.

83. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

84. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

85. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

86. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

87. Use Exercise 86 to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,dx$$

88. (a) If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

(b) Use part (a) to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

5 REVIEW

CONCEPT CHECK

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State both parts of the Fundamental Theorem of Calculus.

4. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

5. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

6. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

7. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

8. State the Substitution Rule. In practice, how do you use it?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then

$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then

$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then

$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. If f' is continuous on $[1, 3]$, then $\int_1^3 f'(v)\,dv = f(3) - f(1)$.

7. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

8. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

9. $\displaystyle\int_{-1}^{1} \left(x^5 - 6x^9 + \frac{\sin x}{(1 + x^4)^2}\right) dx = 0$

10. $\displaystyle\int_{-5}^{5} (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

11. $\displaystyle\int_{-2}^{1} \frac{1}{x^4}\,dx = -\frac{3}{8}$

12. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

13. All continuous functions have derivatives.

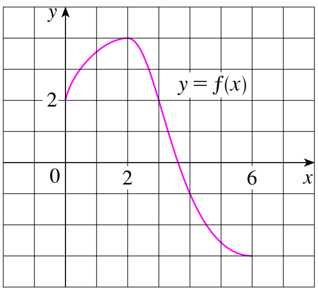
14. All continuous functions have antiderivatives.

15. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

EXERCISES

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \leqslant x \leqslant 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_0^2 (x^2 - x)\,dx$$

(c) Use the Fundamental Theorem to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$

by interpreting it in terms of areas.

4. Express

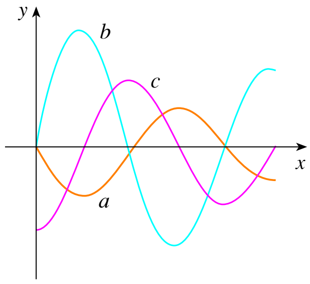
$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin x_i\,\Delta x$$

as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

CAS **6.** (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.
(b) Use the Fundamental Theorem to check your answer to part (a).

7. The following figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$. Identify each graph, and explain your choices.



8. Evaluate:
(a) $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx$ (b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\arctan x}\,dx$
(c) $\displaystyle\frac{d}{dx}\int_0^x e^{\arctan t}\,dt$

9–38 Evaluate the integral, if it exists.

9. $\displaystyle\int_1^2 (8x^3 + 3x^2)\,dx$

10. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx$

11. $\displaystyle\int_0^1 (1 - x^9)\,dx$

12. $\displaystyle\int_0^1 (1 - x)^9\,dx$

13. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

14. $\displaystyle\int_0^1 (\sqrt[4]{u} + 1)^2\,du$

15. $\displaystyle\int_0^1 y(y^2 + 1)^5\,dy$

16. $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\,dy$

17. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

18. $\displaystyle\int_0^1 \sin(3\pi t)\,dt$

19. $\displaystyle\int_0^1 v^2 \cos(v^3)\,dv$

20. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx$

21. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt$

22. $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

23. $\displaystyle\int \left(\frac{1 - x}{x}\right)^2 dx$

24. $\displaystyle\int_1^{10} \frac{x}{x^2 - 4}\,dx$

25. $\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx$

26. $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\,dx$

27. $\displaystyle\int \sin \pi t \cos \pi t\,dt$

28. $\displaystyle\int \sin x \cos(\cos x)\,dx$

29. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

30. $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

31. $\displaystyle\int \tan x \ln(\cos x)\,dx$

32. $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx$

33. $\displaystyle\int \frac{x^3}{1 + x^4}\,dx$

34. $\displaystyle\int \sinh(1 + 4x)\,dx$

35. $\displaystyle\int \frac{\sec\theta \tan\theta}{1 + \sec\theta}\,d\theta$

36. $\displaystyle\int_0^{\pi/4} (1 + \tan t)^3 \sec^2 t\,dt$

37. $\displaystyle\int_0^3 |x^2 - 4|\,dx$

38. $\displaystyle\int_0^4 |\sqrt{x} - 1|\,dx$

39–40 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

39. $\displaystyle\int \frac{\cos x}{\sqrt{1 + \sin x}}\,dx$

40. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 1}}\,dx$

41. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

42. Graph the function $f(x) = \cos^2 x \sin^3 x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

43–48 Find the derivative of the function.

43. $\displaystyle F(x) = \int_0^x \frac{t^2}{1 + t^3}\,dt$

44. $\displaystyle F(x) = \int_x^1 \sqrt{t + \sin t}\,dt$

45. $\displaystyle g(x) = \int_0^{x^4} \cos(t^2)\,dt$

46. $\displaystyle g(x) = \int_1^{\sin x} \frac{1 - t^2}{1 + t^4}\,dt$

47. $\displaystyle y = \int_{\sqrt{x}}^x \frac{e^t}{t}\,dt$

48. $\displaystyle y = \int_{2x}^{3x+1} \sin(t^4)\,dt$

49–50 Use Property 8 of integrals to estimate the value of the integral.

49. $\displaystyle\int_1^3 \sqrt{x^2 + 3}\,dx$

50. $\displaystyle\int_3^5 \frac{1}{x + 1}\,dx$

51–54 Use the properties of integrals to verify the inequality.

51. $\displaystyle\int_0^1 x^2 \cos x\,dx \leqslant \frac{1}{3}$

52. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{2}$

53. $\displaystyle\int_0^1 e^x \cos x\,dx \leqslant e - 1$

54. $\displaystyle\int_0^1 x \sin^{-1} x\,dx \leqslant \pi/4$

55. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3)\,dx$.

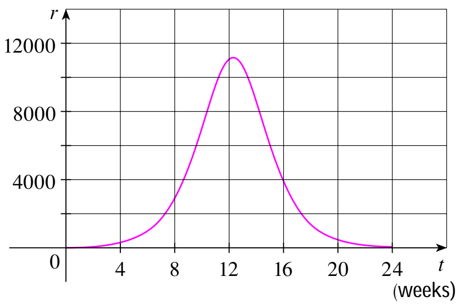
56. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

57. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^8 r(t)\,dt$ represent?

58. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

59. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



60. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leqslant x \leqslant 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$$

Evaluate $\int_{-3}^{1} f(x)\,dx$ by interpreting the integral as a difference of areas.

61. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$.

62. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 5.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

CAS (d) Plot the graphs of C and S on the same screen. How are these graphs related?

63. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

64. Suppose that the temperature in a long, thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\,du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

Use l'Hospital's Rule to find this limit.

65. If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

66. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

67. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x) f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

68. Find $\displaystyle \lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3}\,dt$.

69. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx$$

70. Evaluate

$$\lim_{n \to \infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

71. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\,dy$.