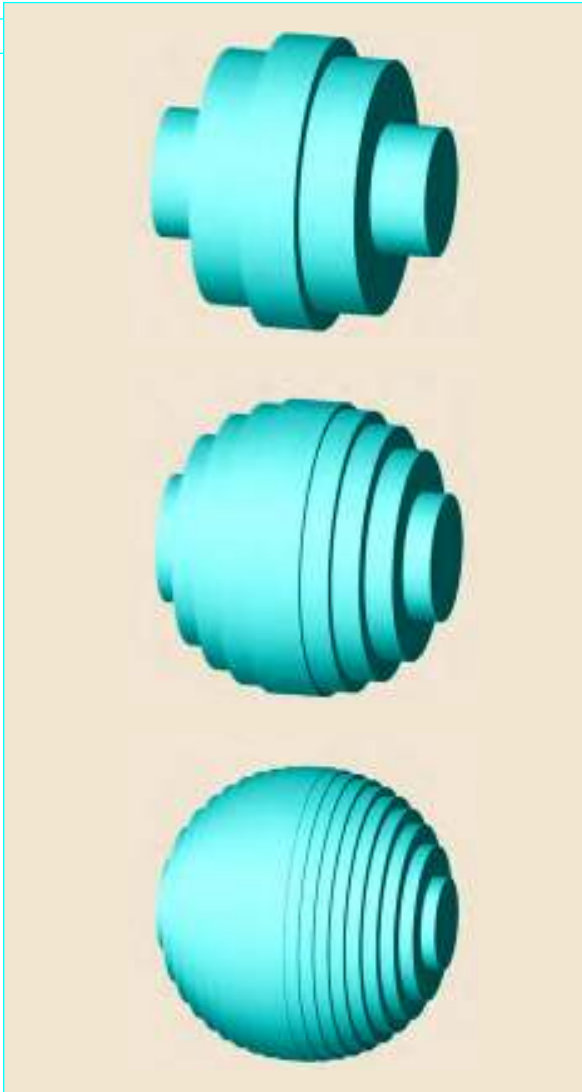# 6

APPLICATIONS OF INTEGRATION

The volume of a sphere is the limit of sums of volumes of approximating cylinders.

In this chapter we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, and the work done by a varying force. The common theme is the following general method, which is similar to the one we used to find areas under curves: We break up a quantity Q into a large number of small parts. We next approximate each small part by a quantity of the form $f(x_i^*) \, \Delta x$ and thus approximate Q by a Riemann sum. Then we take the limit and express Q as an integral. Finally we evaluate the integral using the Fundamental Theorem of Calculus or the Midpoint Rule.

FIGURE 1
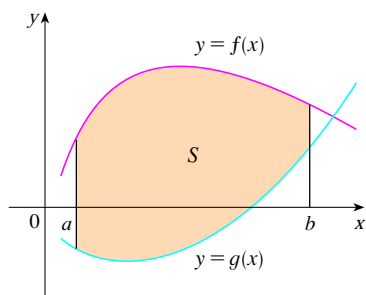$S = \{(x, y) \mid a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)\}$

In Chapter 5 we defined and calculated areas of regions that lie under the graphs of functions. Here we use integrals to find areas of regions that lie between the graphs of two functions.

Consider the region S that lies between two curves $y = f(x)$ and $y = t(x)$ and between the vertical lines $x = a$ and $x = b$, where f and t are continuous functions and $f(x) \geqslant t(x)$ for all x in $[a, b]$. (See Figure 1.)

Just as we did for areas under curves in Section 5.1, we divide S into n strips of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - t(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.) The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - t(x_i^*)] \Delta x$$

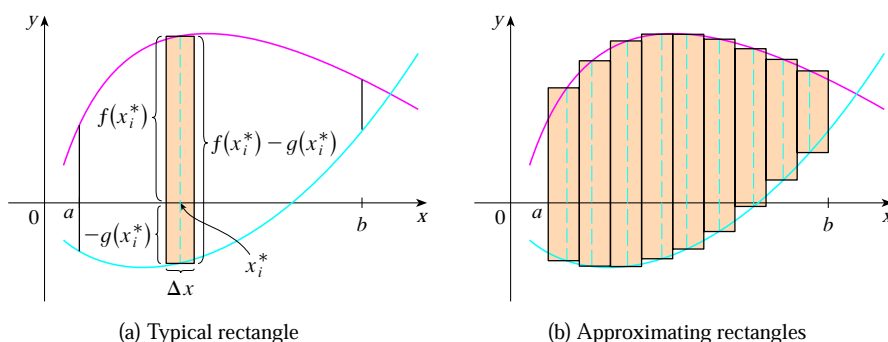is therefore an approximation to what we intuitively think of as the area of S.

FIGURE 2   (a) Typical rectangle   (b) Approximating rectangles

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

$$\boxed{1} \qquad A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - t(x_i^*)] \Delta x$$

We recognize the limit in (1) as the definite integral of $f - g$. Therefore we have the following formula for area.

2 The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and t are continuous and $f(x) \geqslant t(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)] \, dx$$

Notice that in the special case where $t(x) = 0$, S is the region under the graph of f and our general definition of area (1) reduces to our previous definition (Definition 2 in Section 5.1).
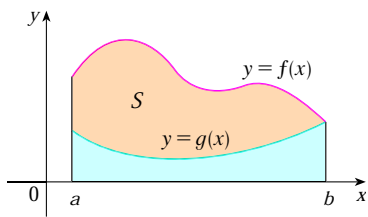
FIGURE 3

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

FIGURE 4

FIGURE 5

FIGURE 6

In the case where both f and t are positive, you can see from Figure 3 why (2) is true:

$$A = [\text{area under } y = f(x)] - [\text{area under } y = g(x)]$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

EXAMPLE 1 Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

SOLUTION The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = e^x$, t$(x) = x$, $a = 0$, and $b = 1$:
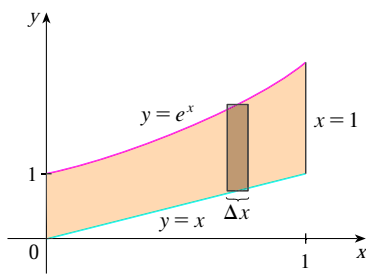
$$A = \int_0^1 (e^x - x)\,dx = e^x - \tfrac{1}{2}x^2 \Big]_0^1$$

$$= e - \tfrac{1}{2} - 1 = e - 1.5 \qquad \blacksquare$$

In Figure 4 we drew a typical approximating rectangle with width Δx as a reminder of the procedure by which the area is defined in (1). In general, when we set up an integral for an area, it's helpful to sketch the region to identify the top curve y_T, the bottom curve y_B, and a typical approximating rectangle as in Figure 5. Then the area of a typical rectangle is $(y_T - y_B)\,\Delta x$ and the equation

$$A = \lim_{n \to \infty} \sum_{i=1}^n (y_T - y_B)\,\Delta x = \int_a^b (y_T - y_B)\,dx$$
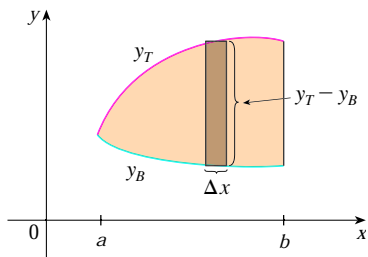
summarizes the procedure of adding (in a limiting sense) the areas of all the typical rectangles.

Notice that in Figure 5 the left-hand boundary reduces to a point, whereas in Figure 3 the right-hand boundary reduces to a point. In the next example both of the side boundaries reduce to a point, so the first step is to find a and b.

✓ EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

SOLUTION We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are $(0, 0)$ and $(1, 1)$.

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

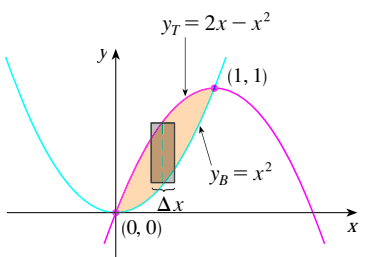and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2 \int_0^1 (x - x^2)\,dx$$

$$= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \qquad \blacksquare$$

Sometimes it's difficult, or even impossible, to find the points of intersection of two curves exactly. As shown in the following example, we can use a graphing calculator or computer to find approximate values for the intersection points and then proceed as before.

EXAMPLE 3 Find the approximate area of the region bounded by the curves $y = x/\sqrt{x^2 + 1}$ and $y = x^4 - x$.

SOLUTION If we were to try to find the exact intersection points, we would have to solve the equation

$$\frac{x}{\sqrt{x^2 + 1}} = x^4 - x$$

This looks like a very difficult equation to solve exactly (in fact, it's impossible), so instead we use a graphing device to draw the graphs of the two curves in Figure 7. One intersection point is the origin. We zoom in toward the other point of intersection and find that $x \approx 1.18$. (If greater accuracy is required, we could use Newton's method or a rootfinder, if available on our graphing device.) Thus an approximation to the area between the curves is

$$A \approx \int_0^{1.18} \left[\frac{x}{\sqrt{x^2 + 1}} - (x^4 - x) \right] dx$$

To integrate the first term we use the substitution $u = x^2 + 1$. Then $du = 2x\,dx$, and when $x = 1.18$, we have $u \approx 2.39$. So

$$A \approx \frac{1}{2} \int_1^{2.39} \frac{du}{\sqrt{u}} - \int_0^{1.18} (x^4 - x)\,dx$$

$$= \sqrt{u}\,\Big]_1^{2.39} - \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^{1.18}$$

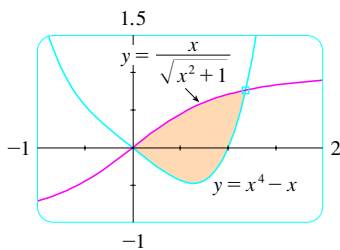$$= \sqrt{2.39} - 1 - \frac{(1.18)^5}{5} + \frac{(1.18)^2}{2}$$
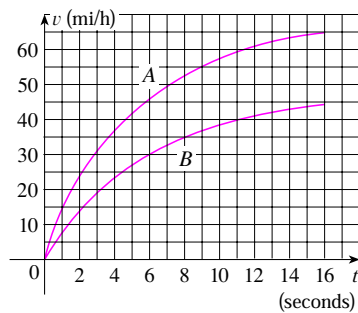
$$\approx 0.785 \qquad \blacksquare$$

EXAMPLE 4 Figure 8 shows velocity curves for two cars, A and B, that start side by side and move along the same road. What does the area between the curves represent? Use the Midpoint Rule to estimate it.

SOLUTION We know from Section 5.4 that the area under the velocity curve A represents the distance traveled by car A during the first 16 seconds. Similarly, the area under curve B is the distance traveled by car B during that time period. So the area between these curves, which is the difference of the areas under the curves, is the distance between the cars after 16 seconds. We read the velocities from the graph and convert them to feet per second (1 mi/h $= \frac{5280}{3600}$ ft/s).

FIGURE 7

FIGURE 8

t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - v_B$	0	13	20	23	25	28	29	29	30

We use the Midpoint Rule with $n = 4$ intervals, so that $\Delta t = 4$. The midpoints of the intervals are $\bar{t}_1 = 2$, $\bar{t}_2 = 6$, $\bar{t}_3 = 10$, and $\bar{t}_4 = 14$. We estimate the distance between the cars after 16 seconds as follows:

$$\int_0^{16} (v_A - v_B)\, dt \approx \Delta t[13 + 23 + 28 + 29]$$

$$= 4(93) = 372 \text{ ft} \qquad \blacksquare$$
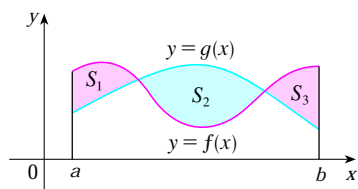
FIGURE 9

If we are asked to find the area between the curves $y = f(x)$ and $y = \mathrm{t}(x)$ where $f(x) \geqslant \mathrm{t}(x)$ for some values of x but $\mathrm{t}(x) \geqslant f(x)$ for other values of x, then we split the given region S into several regions S_1, S_2, . . . with areas A_1, A_2, . . . as shown in Figure 9. We then define the area of the region S to be the sum of the areas of the smaller regions S_1, S_2, . . . , that is, $A = A_1 + A_2 + \cdots$. Since

$$| f(x) - g(x) | = \begin{cases} f(x) - g(x) & \text{when } f(x) \geqslant g(x) \\ g(x) - f(x) & \text{when } g(x) \geqslant f(x) \end{cases}$$

we have the following expression for A.

---

3 The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b | f(x) - g(x) |\, dx$$

When evaluating the integral in (3), however, we must still split it into integrals corresponding to A_1, A_2, ...

EXAMPLE 5 Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

SOLUTION The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 10. Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is
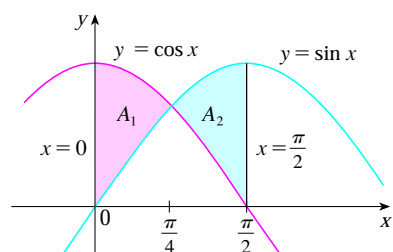

FIGURE 10

$$A = \int_0^{\pi/2} | \cos x - \sin x |\, dx = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x \Big]_0^{\pi/4} + \Big[-\cos x - \sin x \Big]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)$$

$$= 2\sqrt{2} - 2$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2 \int_0^{\pi/4} (\cos x - \sin x)\, dx \qquad \blacksquare$$

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and t are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (see Figure 11), then its area is
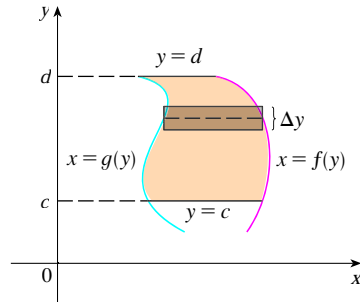
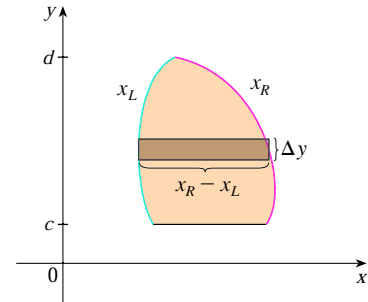$$A = \int_c^d [f(y) - g(y)]\, dy$$

FIGURE 11

FIGURE 12

If we write x_R for the right boundary and x_L for the left boundary, then, as Figure 12 illustrates, we have

$$A = \int_c^d (x_R - x_L)\, dy$$

Here a typical approximating rectangle has dimensions $x_R - x_L$ and Δy.

▼ EXAMPLE 6 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION By solving the two equations we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for x and notice from Figure 13 that the left and right boundary curves are

$$x_L = \tfrac{1}{2} y^2 - 3 \qquad x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^4 (x_R - x_L)\, dy$$

$$= \int_{-2}^4 \left[(y + 1) - \left(\tfrac{1}{2} y^2 - 3 \right) \right] dy$$

$$= \int_{-2}^4 \left(-\tfrac{1}{2} y^2 + y + 4 \right) dy$$

$$= -\frac{1}{2} \left(\frac{y^3}{3} \right) + \frac{y^2}{2} + 4y \Bigg]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8 \right) = 18 \qquad ■$$

FIGURE 13

FIGURE 14

We could have found the area in Example 6 by integrating with respect to x instead of y, but the calculation is much more involved. It would have meant splitting the region in two and computing the areas labeled A_1 and A_2 in Figure 14. The method we used in Example 6 is *much* easier.
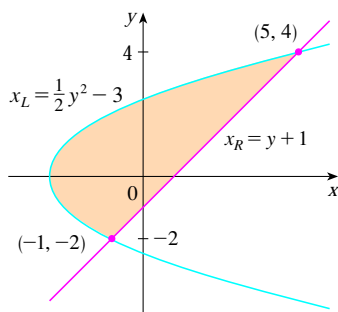
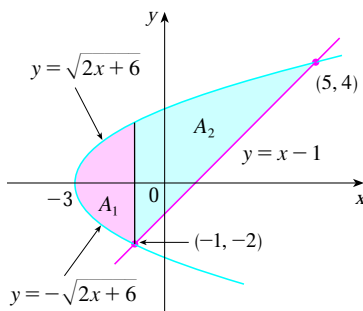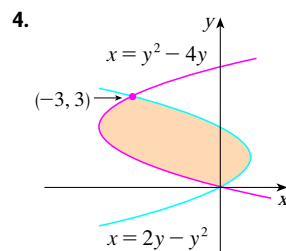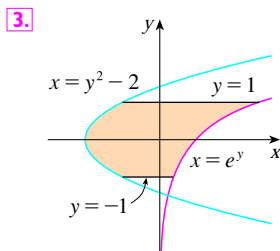## 6.1 EXERCISES

1–4 Find the area of the shaded region.

1.

$y = 5x - x^2$
$(4, 4)$
$y = x$

2.

$y = \sqrt{x+2}$
$x = 2$
$y = \dfrac{1}{x+1}$

3.

$x = y^2 - 2$
$y = 1$
$x = e^y$
$y = -1$

4.

$x = y^2 - 4y$
$(-3, 3)$
$x = 2y - y^2$
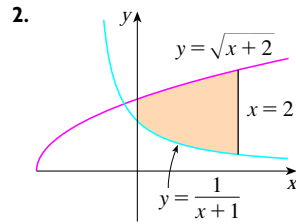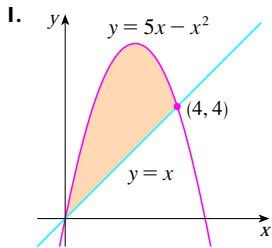
5–28 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

5. $y = x + 1$, $y = 9 - x^2$, $x = -1$, $x = 2$

6. $y = \sin x$, $y = e^x$, $x = 0$, $x = \pi/2$

7. $y = x$, $y = x^2$

8. $y = x^2 - 2x$, $y = x + 4$

9. $y = 1/x$, $y = 1/x^2$, $x = 2$

10. $y = 1 + \sqrt{x}$, $y = (3 + x)/3$

11. $y = x^2$, $y^2 = x$

12. $y = x^2$, $y = 4x - x^2$

13. $y = 12 - x^2$, $y = x^2 - 6$

14. $y = \cos x$, $y = 2 - \cos x$, $0 \le x \le 2\pi$

15. $y = \tan x$, $y = 2 \sin x$, $-\pi/3 \le x \le \pi/3$

16. $y = x^3 - x$, $y = 3x$

17. $y = \sqrt{x}$, $y = \frac{1}{2}x$, $x = 9$

18. $y = 8 - x^2$, $y = x^2$, $x = -3$, $x = 3$

19. $x = 2y^2$, $x = 4 + y^2$

20. $4x + y^2 = 12$, $x = y$

21. $x = 1 - y^2$, $x = y^2 - 1$

22. $y = \sin(\pi x/2)$, $y = x$

23. $y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \pi/2$

24. $y = \cos x$, $y = 1 - \cos x$, $0 \le x \le \pi$

25. $y = x^2$, $y = 2/(x^2 + 1)$

26. $y = |x|$, $y = x^2 - 2$

27. $y = 1/x$, $y = x$, $y = \frac{1}{4}x$, $x > 0$

28. $y = 3x^2$, $y = 8x^2$, $4x + y = 4$, $x \ge 0$

29–30 Use calculus to find the area of the triangle with the given vertices.

29. $(0, 0)$, $(2, 1)$, $(-1, 6)$

30. $(0, 5)$, $(2, -2)$, $(5, 1)$

31–32 Evaluate the integral and interpret it as the area of a region. Sketch the region.

31. $\displaystyle\int_0^{\pi/2} |\sin x - \cos 2x| \, dx$

32. $\displaystyle\int_0^4 |\sqrt{x + 2} - x| \, dx$

33–34 Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the given curves.

33. $y = \sin^2(\pi x/4)$, $y = \cos^2(\pi x/4)$, $0 \le x \le 1$

34. $y = \sqrt[3]{16 - x^3}$, $y = x$, $x = 0$

35–38 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

**35.** $y = x \sin(x^2)$, $y = x^4$

36. $y = e^x$, $y = 2 - x^2$

37. $y = 3x^2 - 2x$, $y = x^3 - 3x + 4$

38. $y = x \cos x$, $y = x^{10}$

CAS **39.** Use a computer algebra system to find the exact area enclosed by the curves $y = x^5 - 6x^3 + 4x$ and $y = x$.

40. Sketch the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ and find its area.

41. Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in miles per hour) during the first ten seconds of the race. Use the Midpoint Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.

t	v_C	v_K	t	v_C	v_K
0	0	0	6	69	80
1	20	22	7	75	86
2	32	37	8	81	93
3	46	52	9	86	98
4	54	61	10	90	102
5	62	71			

42. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use the Midpoint Rule to estimate the area of the pool.
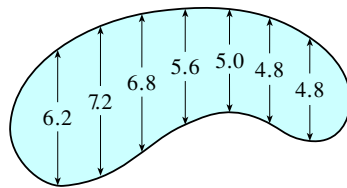
43. A cross-section of an airplane wing is shown. Measurements of the height of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing's cross-section.
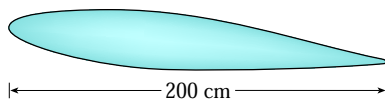
44. If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \leq t \leq 10$. What does this area represent?

45. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.
(a) Which car is ahead after one minute? Explain.
(b) What is the meaning of the area of the shaded region?

(c) Which car is ahead after two minutes? Explain.
(d) Estimate the time at which the cars are again side by side.
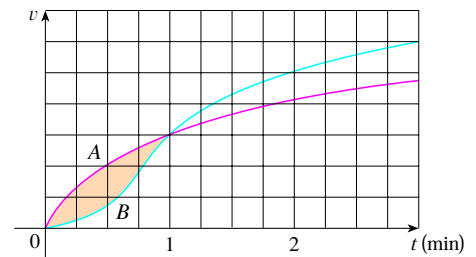
46. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 4.7 that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.
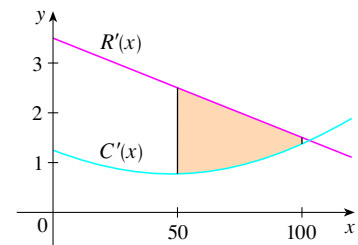
47. The curve with equation $y^2 = x^2(x + 3)$ is called **Tschirnhausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

48. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

49. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

50. (a) Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \leq x \leq 4$.
(b) Find the number b such that the line $y = b$ bisects the area in part (a).

51. Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

52. Suppose that $0 < c < \pi/2$. For what value of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$, and $y = 0$?

53. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.